

MATH 340: GRAPH THEORY REVIEW.

BASIC NOTIONS. REVIEW OF MATH 240.

Definition of a graph. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of *vertices*, $E(G)$ is a set of some pairs of vertices called *edges*.¹ We will write uv , instead of $\{u, v\}$, to denote the edge consisting of a pair of vertices u and v for brevity. For example, if

$$V(G) = \{a, b, c, d\},$$
$$E(G) = \{ab, ac, bc, cd\},$$

then G is a graph with four vertices and four edges.

The vertices u and v are called *the ends* of the edge $e = uv$, and the edge e is *incident* to its ends. Two vertices u and v are *adjacent* or *neighbors* if $uv \in E(G)$.

Standard graph classes. A *complete graph* on n vertices is denoted K_n is a simple graph in which every two vertices are adjacent.

A *path* on n vertices, denoted P_n , is a graph such that:

$$V(P_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}.$$

The vertices v_1 and v_n are *the ends* of the path P_n .

A *cycle* on $n \geq 3$ vertices, denoted C_n , is a graph such that:

$$V(C_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}.$$

Degrees. The *degree* of a vertex v in a graph G is the number of its neighbors. It is denoted by $\deg_G(v)$, or usually simply $\deg(v)$ when the graph G is understood from context.

Theorem 1. For any graph G , we have:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Subgraphs. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ to denote that H is a subgraph of G .

A *path* (or *cycle*) of G is a subgraph of G which is a path (or resp. cycle).

A subgraph $G \setminus e$ obtained from G by *deleting the edge* $e \in E(G)$ is defined by $V(G \setminus e) = V(G)$, $E(G \setminus e) = E(G) - \{e\}$. A subgraph $G \setminus v$ of G obtained from G by *deleting the vertex* v is defined by $V(G \setminus v) = V(G) - \{v\}$, and $E(G \setminus v)$ consists of all the edges of G not incident to v .

Isomorphism. Graphs H and G are *isomorphic* if there exists a bijection $\phi : V(H) \rightarrow V(G)$ (called an *isomorphism*) such that $uv \in E(H)$ if and only if $\phi(u)\phi(v) \in E(G)$. We frequently treat isomorphic graphs as being the same.

¹Unlike other definitions this one does not allow for loops and parallel edges.

Connectivity. A *walk* from v_0 to v_k in a graph G is a non-empty alternating sequence $v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k$ of vertices and edges in G , such that $e_i = v_i v_{i+1}$ for $i = 0, \dots, k-1$. If $v_0 = v_k$, then the walk is said to be *closed*.

A graph G is *connected* if and only if for all $u, v \in V(G)$, there exists a walk from u to v .

Lemma 2. *If there is a walk with ends u, v in G , then there is a path in G with the same ends.*

A *connected component* of a graph G is a maximal connected subgraph.

Lemma 3. *In a graph G , every vertex is in a unique connected component.*

Let $\text{comp}(G)$ denote the number of components of a graph G .

Trees and Forests. A *forest* is a graph with no cycles. A *tree* is a connected forest.²

Theorem 4. *If G is a (non-null) forest, then*

$$\text{comp}(G) = |V(G)| - |E(G)|.$$

In particular, if G is a tree then

$$|V(T)| = |E(T)| + 1$$

A vertex v in a tree with $\deg(v) = 1$ is called a *leaf*.

Lemma 5. *Let T be a tree with $|V(T)| \geq 2$. Then, T has at least two leaves, and if T has exactly two leaves, then T is a path.*

Bipartite Graphs. A *bipartition* of a graph G is a pair of subsets (A, B) of $V(G)$ so that $A \cap B = \emptyset$, $A \cup B = V(G)$, and every edge of G has one end in A and another in B . A graph is *bipartite* if it admits a bipartition.

Theorem 6. *For every graph G , the following statements are equivalent:*

- (1) G is bipartite,
- (2) G has no closed walk with odd number of edges,
- (3) G has no odd cycle.

Vertex coloring. Let G be a graph, S a set of size k . The function $\varphi : V(G) \rightarrow S$ is called a (*proper*) k -*coloring* if for all $e \in E(G)$ with ends u and v , $\varphi(u) \neq \varphi(v)$. Elements of S are called *colors*. The set of all vertices of the same color is called a *color class*.

The *chromatic number* $\chi(G)$ is the minimum k such that G admits a k -coloring.

Let $\Delta(G)$ denote the maximum degree of a vertex in a graph G .

Theorem 7. $\chi(G) \leq \Delta(G) + 1$ for every graph G .

²If one is being pedantic, then one should note that the *null graph* with no vertices and no edges is considered to be a forest, but not a tree.

MATCHINGS

Stable Matchings. Given a group of n men and n women, a matching M which pairs each man with exactly one woman is *stable* if there does not exist a pair a man b and a woman g not paired by M such that b and g prefer each other to their current partners.

Boy proposal algorithm: At each step, each man proposes to the woman he likes most, and which has not rejected him yet. Each woman rejects all men who propose to her, except for the one among them she likes the most. To this one, she says maybe. If no woman receives > 1 proposal, every woman marries the man proposing to her, thus creating a matching.

Theorem 8. *The boy proposal algorithm produces a stable matching.*

Matchings in bipartite graphs. A *matching* $M \subset E(G)$ in a graph G is a collection of edges so that every vertex of G is incident to at most one edge of M . A matching M is *perfect* if every vertex of G is incident to exactly one edge of G .

Theorem 9 (Hall). *Let G be bipartite, with bipartition (A, B) . Then, the following are equivalent:*

- (1) *There exists a matching in G covering A .*
- (2) *For every $S \subset A$, vertices in S have at least $|S|$ neighbours in B , that is $|N(S)| \geq |S|$.*

Corollary 10. *Let G be bipartite, with every vertex of the same degree $d > 0$. Then, G has a perfect matching.*

Let $\nu(G)$, the *matching number* of G , denote the maximum size of a matching in G .

A subset $X \subset V(G)$ is a *vertex cover* of G if every edge of G has an end in X . Let $\tau(G)$, the *vertex cover number* of G , denote the minimum size of a vertex cover in G .

Theorem 11 (König). *If G is bipartite, then $\nu(G) = \tau(G)$.*

Edge coloring. A *k -edge colouring* of a graph G is a map $\varphi : E(G) \rightarrow S$ with $|S| = k$, so that $\varphi(e) \neq \varphi(f)$ if e and f share an end.

The minimum k such that G admits a k -edge colouring is called the *edge-chromatic number* of G and is denoted $\chi'(G)$.

Theorem 12. $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.

Theorem 13 (König). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

PLANAR GRAPHS

A *drawing* of a graph G in the plane represents vertices as points in the plane, and edges as curves which do not intersect themselves or each other, and have ends at points corresponding to the ends of edges. Points in the plane not used in the drawing are divided into *regions*. Two points belong to the same region if they can be joined by a curve which avoids the drawing.

Let $\text{Reg}(G)$ denote the number of regions in the planar drawing of G .

Theorem 14 (Euler's Formula). *Let G be a connected graph drawn in the plane. Then*

$$|V(G)| - |E(G)| + \text{Reg}(G) = 2.$$

Lemma 15. *If G is planar, $|E(G)| \geq 2$, then $|E(G)| \leq 3|V(G)| - 6$. If, further, G contains no K_3 subgraphs, then $|E(G)| \leq 2|V(G)| - 4$.*

Corollary 16. *The graphs K_5 and $K_{3,3}$ are non-planar.*

Theorem 17 (The Four Color Theorem). $\chi(G) \leq 4$ for every planar graph G .

Theorem 18 (Fary). *Every planar graph can be drawn in the plane so that the edges are represented by the straight lines.*

Minors. *Contracting an edge e with ends u and v means deleting e and identifying u and v . A graph H is a *minor* of a graph G if it can be obtained from G by repeatedly deleting edges and vertices and contracting edges.*

Being a minor is a transitive relation, i.e. if H' is a minor of H , and H is a minor of G , then H' is a minor of G .

Conjecture 19 (Hadwiger). *If G does not have a K_t minor, then $\chi(G) \leq t - 1$.*

A graph G is a *subdivision* of H if edges of H are replaced in G by internally disjoint paths. If G is a subdivision of H , then H is a minor of G .

Theorem 20 (Kuratowski). (1) *A graph G is planar if and only if G does not contain either K_5 or $K_{3,3}$ as a minor.*

(2) *A graph G is planar if and only if G does not contain a subdivision of either K_5 or $K_{3,3}$ as a subgraph.*